SACE BOARD
SOUTH AUSTRALIA

Stage 2 Specialist Mathematics

Sample examination questions - 1

Government
of South Australia

PART 1

Question 1 (6 marks)

Consider the curve with equation $2x^2 - 4xy + 16y^2 = 7$, as shown in Figure 1.

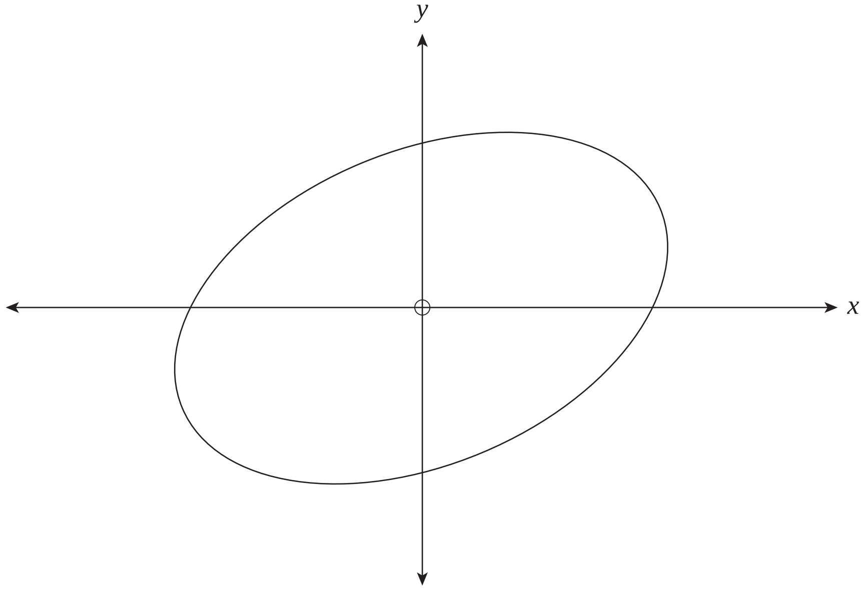

Figure 1

(a) Use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x}{8y - x}$ where $8y \neq x$.

(3 marks)

(b) Hence find the exact coordinates of *all* points on the curve where the tangent to the curve is horizontal.

(3 marks)

Question 2 (7 marks)

Consider the polynomial $P(x) = kx^4 + ax^2 + bx + c$, where k, a, b, and c are real constants.

The polynomial $P(x)$ has zeros $x = -1$ and $x = 1$, and when $P(x)$ is divided by $(x-2)$ the remainder is 4.

(a) Show that an augmented matrix for the coefficients a, b, and c can be written as:

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & -k \\ 1 & 1 & 1 & -k \\ 4 & 2 & 1 & 4-16k \end{array}\right].$$

(2 marks)

(b) Find a, b, and c in terms of k, clearly stating all row operations.

(3 marks)

(c) Find the polynomial $P(x)$ when $k=-1$.

(2 marks)

Question 3 (6 marks)

(a) (i) Write $\sqrt{2}+\mathrm{i}\sqrt{2}$ in polar form.

(1 mark)

(ii) Write $\sqrt{2}-\mathrm{i}\sqrt{2}$ in polar form.

(1 mark)

(b) (i) Write $z=\dfrac{\left(\sqrt{2}+\mathrm{i}\sqrt{2}\right)^m}{\left(\sqrt{2}-\mathrm{i}\sqrt{2}\right)^n}$ in simplest polar form, where m and n are integers.

(2 marks)

(ii) State a positive integer value for m and a positive integer value for n such that z is real.

(1 mark)

(iii) State a positive integer value for m and a positive integer value for n such that z is purely imaginary.

(1 mark)

Question 4 (6 marks)

(a) If the vector $\boldsymbol{p} = [p_1, p_2]$, show that $\boldsymbol{p} \bullet \boldsymbol{p} = |\boldsymbol{p}|^2$.

(1 mark)

(b) Triangle ABC is isosceles with sides AB and CB of equal length. A line is drawn from B to meet AC at right angles at point O, as shown in Figure 2.

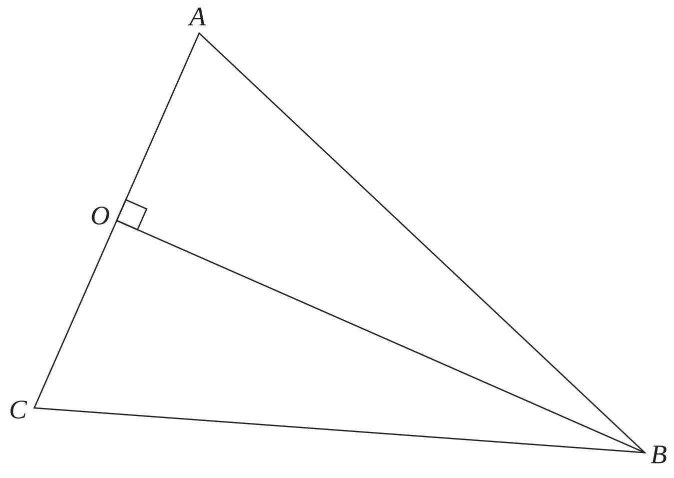

Figure 2

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, and $\overrightarrow{OC} = \boldsymbol{c}$.

(i) Find the value of $\boldsymbol{a} \bullet \boldsymbol{b}$.

(1 mark)

(ii) Write $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(1 mark)

(iii) Show that $|\boldsymbol{b}-\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{a}|^2$.

(1 mark)

(iv) Using the fact that $|\overrightarrow{AB}| = |\overrightarrow{CB}|$, show that O is the mid-point of AC.

(2 marks)

Question 5 (8 marks)

In a compost heap, organic material is left to decompose. As the material decomposes, the mass of the compost heap changes. The rate of change of the mass of the compost heap can be modelled by the differential equation

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$$\frac{\mathrm{d}M}{\mathrm{d}t} = -0.02M$$

where M is the mass of the material in kilograms and t is the time in days.

(a) If $M = 30$ at $t = 0$, solve the differential equation.

(3 marks)

If a compost heap is not treated with air and water, the temperature within the compost heap will increase.

The rate of change of the temperature within a compost heap that is not treated with air and water can be modelled by the differential equation

$$\frac{dT}{dt} = 1.25e^{-0.05t}$$

where T is the temperature within the compost heap in degrees Celsius (°C) and t is the time in days.

(b) If $T = 22$ at $t = 0$, solve the differential equation to show that $T = -25e^{-0.05t} + 47$.

(3 marks)

(c) If the temperature within the compost heap increases to $40°\text{C}$, a fire may start within the compost heap.

(i) At what time, t, is the temperature within the compost heap $40°\text{C}$?

(1 mark)

(ii) Find the mass of the compost heap at this time.

(1 mark)

Question 6 (8 marks)

(a) Use mathematical induction to prove that

$$(1+3)\left(1+3^2\right)\left(1+3^4\right)\ \dots\ \left(1+3^{2^{n-1}}\right)=\frac{3^{2^n}-1}{2}$$

for all positive integers n.

(6 marks)

(b) Hence simplify $(1+3^4)(1+3^8)\ \ldots\ (1+3^{128})$.

(2 marks)

Question 7 (7 marks)

A cone is formed as grain is poured into a large cylindrical container of fixed radius.

The volume of the cone of grain in cubic metres is given by

$$V = \frac{1}{3}\pi h^3 \tan^2 \theta$$

where h is the height of the cone in metres and θ (in radians) is the angle shown in Figure 3.

Note that V, h, and θ are all functions of time.

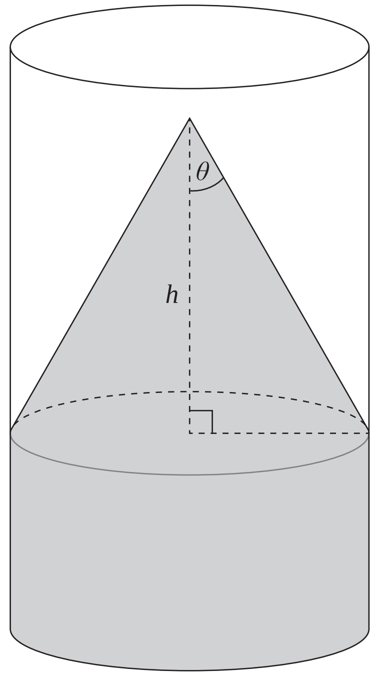

Figure 3

(a) Show that $\frac{dV}{dt} = \frac{1}{3}\pi h^2 \left(2h \tan\theta \sec^2\theta \frac{d\theta}{dt} + 3\frac{dh}{dt}\tan^2\theta \right)$.

(3 marks)

(b) Consider the instant when $\theta = \frac{\pi}{6}$ radians and $V = 3\pi$ cubic metres.

(i) Find h at this instant.

(1 mark)

(ii) At this instant, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\pi}{12}$ radians per second and the height of the cone is decreasing at approximately 1.81 metres per second.

Find $\frac{\mathrm{d}V}{\mathrm{d}t}$.

(3 marks)

Question 8 (8 marks)

(a) Find the distance between the point with coordinates $(6, 3, -4)$ and the plane with equation $2x + y - 2z = 6$.

(2 marks)

(b) (i) For real numbers a, b, c, r, and q, where $a \neq 0$, show that:

(1) the point $P\left(\frac{r}{a}, 0, 0\right)$ is on the plane with equation $ax + by + cz = r$.

(1 mark)

(2) the distance between P and the plane with equation $ax + by + cz = q$ is

$$\frac{|r - q|}{\sqrt{a^2 + b^2 + c^2}}.$$

(2 marks)

(ii) Find the distance between the plane with equation $ax+by+cz=r$ and the plane with equation $ax+by+cz=q$.

(1 mark)

(c) If the distance between the plane with equation $2x+y-2z=6$ and the plane with equation $2x+y-2z=k$ is 12, find the possible values of k.

(2 marks)

Question 9 (9 marks)

(a) On the axes in Figure 4, sketch the graph of $f(x) = \frac{2}{(x-1)(x+1)}$.

Clearly show the behaviour of the function near the asymptotes.

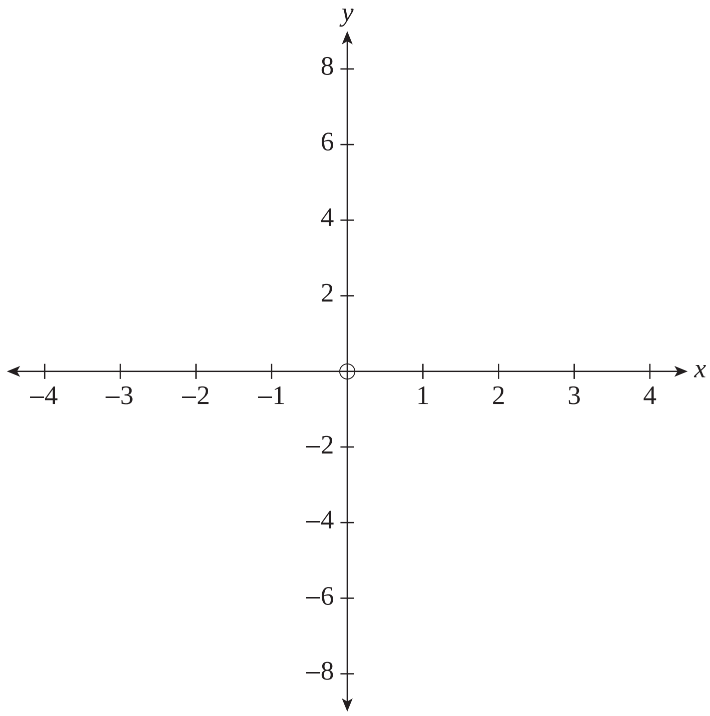

Figure 4

(3 marks)

(b) Show that $\frac{2}{(x-1)(x+1)} = \frac{1}{(x-1)} - \frac{1}{(x+1)}$.

(1 mark)

(c) Hence show that $\left(f(x)\right)^2 = \frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \frac{1}{(x-1)} + \frac{1}{(x+1)}$.

(1 mark)

(d) Find the *exact* volume of the solid that is obtained when the region bounded by the graph of $f(x)$ and the x-axis on the interval $[2, 3]$ is rotated about the x-axis.

(4 marks)

Question 10 (10 marks)

Figure 5 shows part of a circle of radius 52 in the complex plane. The centre of the circle is at $c = -47 + 21\mathrm{i}$.

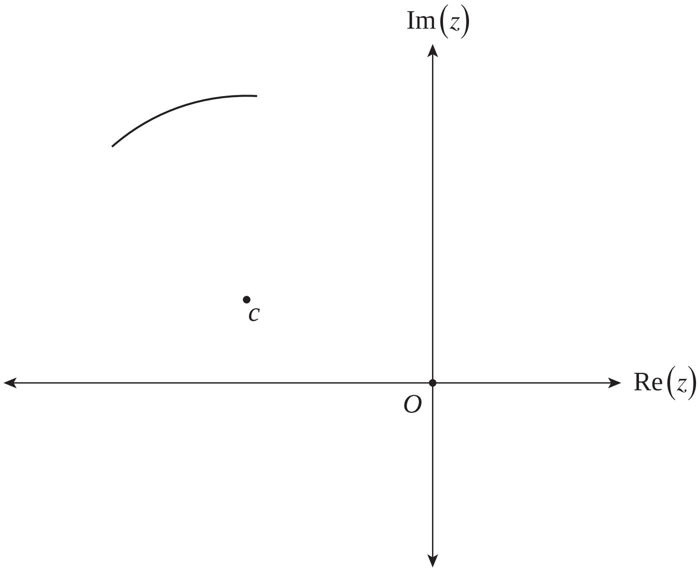

Figure 5

(a) Write down an equation in terms of z that describes *exactly* all complex numbers that lie on the circle.

(2 marks)

(b) (i) Calculate $|c|$ correct to three decimal places.

(1 mark)

(ii) Explain why the circle, if drawn completely, will cut the positive real axis.

(2 marks)

(c) Suppose the circle cuts the positive real axis at w.

Use the triangle inequality for the triangle with vertices at O, c, and w to show that $|w| > 0.52$.

(3 marks)

(d) Use the equation of the circle to find the value of w.

(2 marks)

Question 11 (15 marks)

(a) One fish tank contains 30 fish. The growth rate of this population of fish can be modelled by

$$\frac{dP}{dt} = kP\left(\frac{270 - P}{270}\right)$$

where P is the number of fish, t is time in days, and k is a real constant.

(i) On the slope field in Figure 6, draw the solution curve for P starting at the point $(0, 30)$.

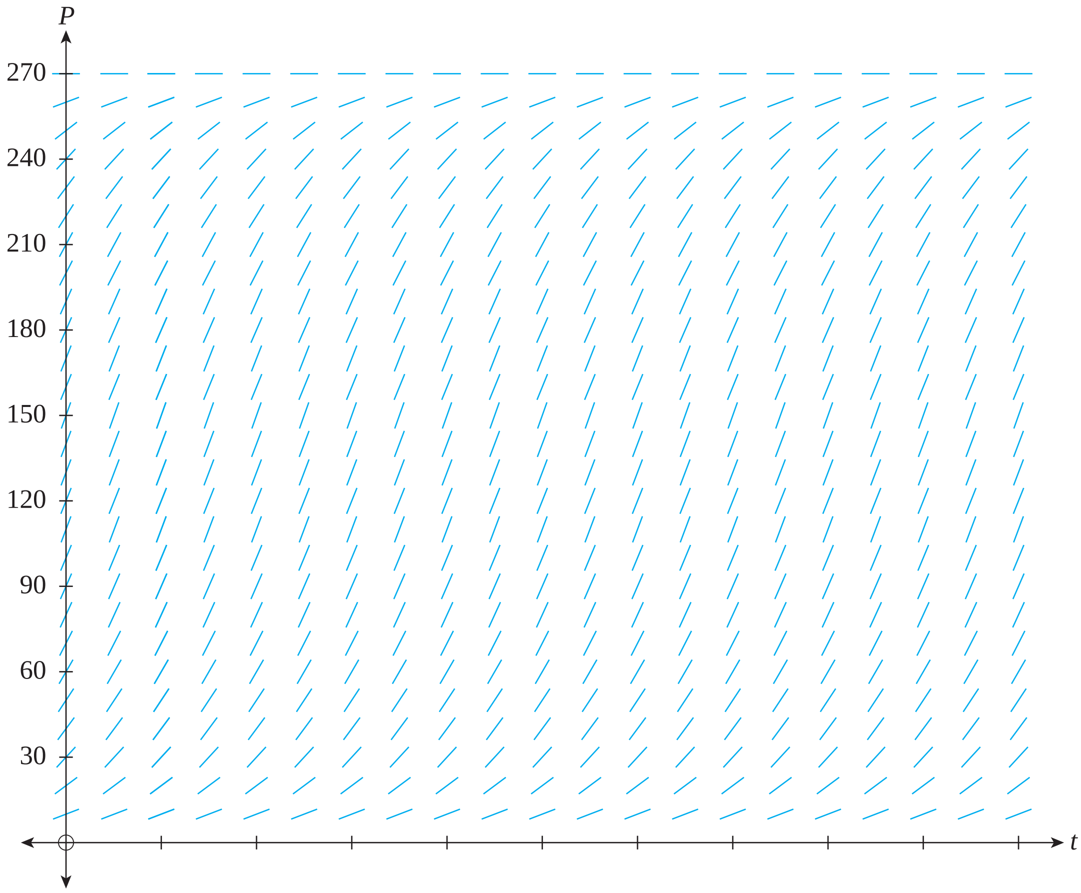

Figure 6

(3 marks)

(ii) Show that $\dfrac{1}{P}+\dfrac{1}{270-P}=\dfrac{270}{P(270-P)}$.

(1 mark)

(iii) Initially there are 30 fish in the fish tank. Use integration to solve the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(\frac{270-P}{270}\right)$$

and show that $P=\dfrac{270}{1+8e^{-kt}}$.

(5 marks)

(b) At the same time, a different population of fish in another fish tank is observed.

The growth rate of this population is modelled by the differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = (B-30)\left(\frac{1-0.4t}{t}\right)$$

where B is the number of fish and t is time in days.

(i) If $B = 30\left(5e^{-2}+1\right)$ when $t = 5$, use integration to solve the differential equation and show that

$$B = 30\left(te^{-0.4t}+1\right).$$

(4 marks)

(ii) The equation for population P is $P = \dfrac{270}{1+8e^{-kt}}$.

Find the limiting size of population P and population B as $t \to \infty$.

(2 marks)

Question 12 (15 marks)

Figure 7 shows the points $A(2,1,4)$, $B(4,-1,0)$, and $C(5,-1,2)$ on the plane P_1.

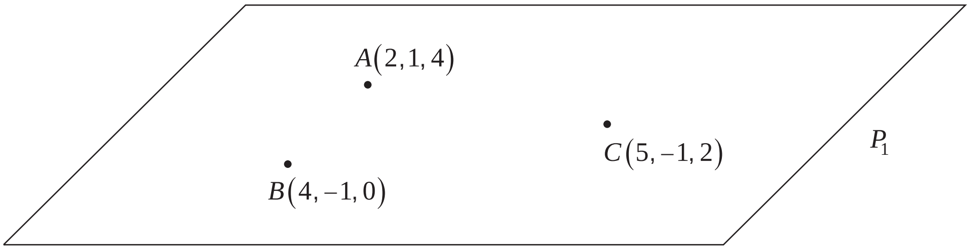

Figure 7

(a) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(2 marks)

(ii) Find the area of triangle ABC.

(2 marks)

(iii) Show that the equation of P_1 is $2x+4y-z=4$.

(2 marks)

(b) (i) The equation of P_1 is $2x+4y-z=4$.

Show that the parametric equations of the normal to P_1 through $A(2,1,4)$ are

$$\begin{cases} x=2+2t \\ y=1+4t \\ z=4-t \end{cases} \quad \text{where } t \text{ is a parameter.}$$

(1 mark)

(ii) P_2 is a plane with equation $2x+4y-z=25$.

Show that the normal to P_1 through A intersects P_2 at $A'(4,5,3)$.

(2 marks)

(iii) Find where the normals to P_1 through B and C intersect P_2. Name these points B' and C' respectively.

(2 marks)

(iv) Find the area of triangle $A'B'C'$.

(1 mark)

(c) P_3 is a plane with equation $2x - z = 25$. The normals to P_1 through A, B, and C intersect P_3 at points A'', B'', and C'' respectively.

Is the area of triangle $A''B''C''$ the same as, less than, or greater than the area of triangle ABC? Explain your answer.

(3 marks)

Question 13 (15 marks)

(a) (i) Solve the equation $z^3 = 1$. Write the three solutions in polar form.

(2 marks)

(ii) Draw the solutions on the Argand diagram in Figure 8, labelling each solution in an anticlockwise direction α_1, α_2, and α_3, where α_1 is real.

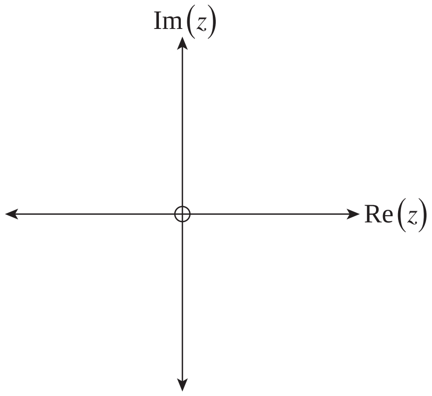

Figure 8

(2 marks)

(iii) Explain why α_2 lies on the line with equation $|z - \alpha_1| = |z - \alpha_3|$.

(1 mark)

(iv) Evaluate $\alpha_1 + \alpha_2 + \alpha_3$.

(1 mark)

(v) Evaluate $|\alpha_1|+|\alpha_2|+|\alpha_3|$.

(1 mark)

(vi) Find the *exact* value of $|\alpha_2-\alpha_3|$.

(1 mark)

(b) Consider the equation $(w-1)^3=1$.

(i) Show that the solutions to this equation are $w_1=1+\alpha_1$, $w_2=1+\alpha_2$, and $w_3=1+\alpha_3$, where α_1, α_2, and α_3 are the solutions to the equation $z^3=1$, as in part (a).

(1 mark)

(ii) Draw the solutions of $(w-1)^3=1$ on the Argand diagram in Figure 9.

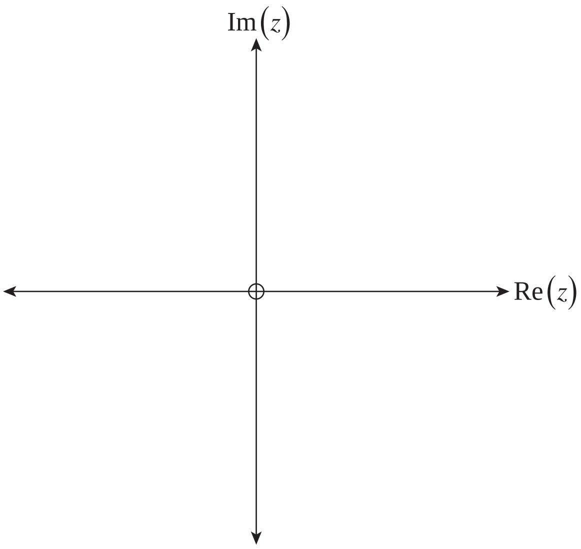

Figure 9

(2 marks)

(iii) Show that w_2 and w_3 lie on the line $\mathrm{Re}(z) = \frac{1}{2}$.

(1 mark)

(iv) Evaluate $|w_1| + |w_2| + |w_3|$.

(1 mark)

(v) Find the *exact* value of $|w_1 - w_2| + |w_2 - w_3| + |w_3 - w_1|$.

(2 marks)

Question 14 (15 marks)

(a) Figure 10 shows the graph of the function $f(x) = \arccos\left(\frac{x}{2}\right)$ for $-2 \le x \le 2$.

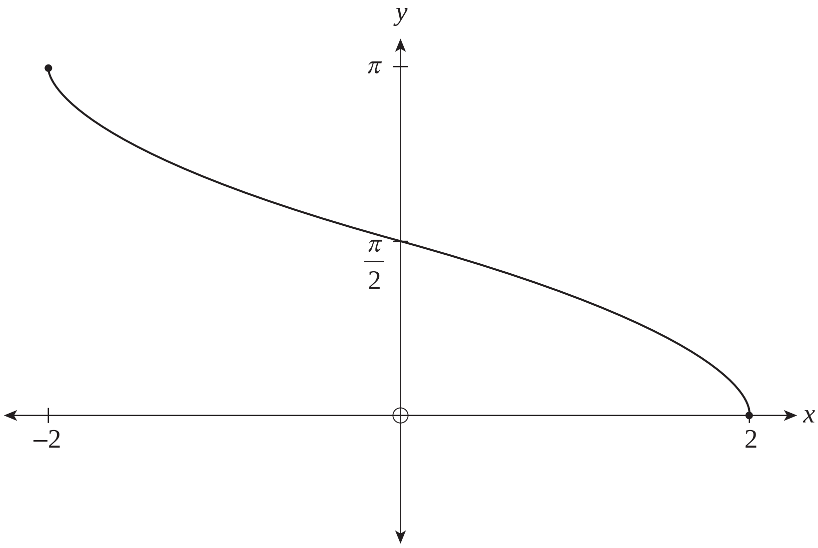

Figure 10

(i) State why $f(x)$ for $-2 \le x \le 2$ has an inverse function.

(1 mark)

(ii) Find the equation of $f^{-1}(x)$.

(2 marks)

(iii) State the range of $f^{-1}(x)$.

(1 mark)

(b) If $y = \arccos\left(\frac{x}{2}\right)$ then $\frac{x}{2} = \cos y$.

Hence use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\sqrt{4-x^2}}$.

(3 marks)

(c) Find $\int \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x$.

(2 marks)

(d) (i) Use integration by parts to show that

$$\int \arccos\left(\frac{x}{2}\right)\mathrm{d}x = x\arccos\left(\frac{x}{2}\right) - \sqrt{4-x^2} + c,$$ where c is a constant.

(3 marks)

(ii) Hence find the *exact* area between the graph of $y = \arccos\left(\frac{x}{2}\right)$, the x-axis, and the lines $x = -2$ and $x = 2$.

(3 marks)

Question 15 (15 marks)

The following parametric equations describe the path of an ice-skater:

$$\begin{cases} x(t) = 3\cos t \\ y(t) = 4\sin 2t \end{cases}$$

where the parameter t is time and $0 \le t \le 2\pi$.

(a) On the axes in Figure 11, draw the parametric curve that describes the path of the ice-skater.

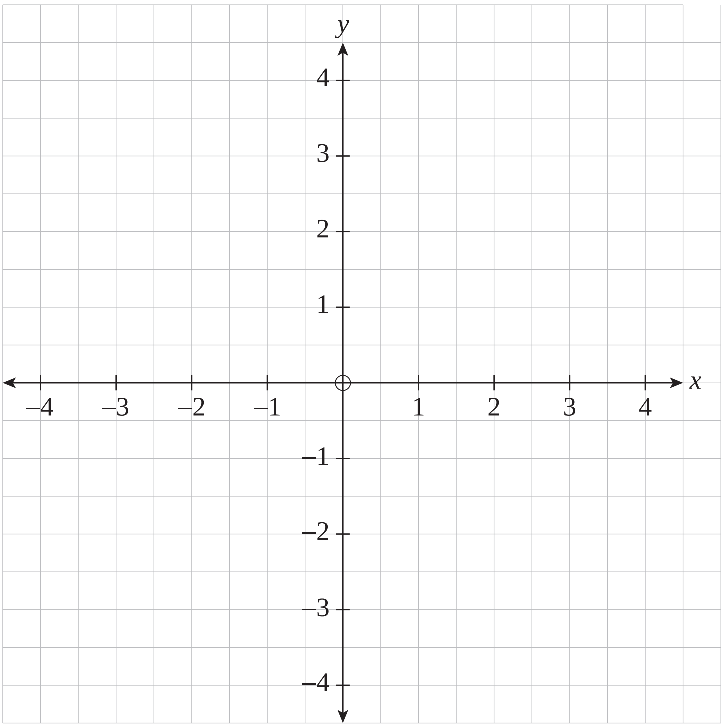

Figure 11

(3 marks)

(b) (i) Find the velocity vector $\mathbf{v}$ of the ice-skater at time t.

(2 marks)

(ii) On the curve that you drew in part (a), draw the velocity vector at $t = \frac{\pi}{4}$. (2 marks)

(c) The ice-skater skates through position P_1 at $t = \frac{\pi}{2}$ and position P_2 at $t = \frac{5\pi}{4}$.

(i) On the curve that you drew in part (a), mark P_1 and P_2. (1 mark)

(ii) Show that the speed of the ice-skater is given by $S = \sqrt{9\sin^2 t + 64\cos^2 2t}$.

(1 mark)

(iii) Hence find the distance that the ice-skater travels while skating from P_1 to P_2.

(2 marks)

(iv) Find the maximum speed and the minimum speed of the ice-skater, and the times at which these occur.

(4 marks)