SACE BOARD
SOUTH AUSTRALIA

Stage 2 Mathematical Methods

Sample examination questions - 2

Government
of South Australia

Question 1 (8 marks)

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following. There is no need to simplify your answers.

(a) $y = x^2 e^x$.

(2 marks)

(b) $y = (x + \cos x)^4$.

(3 marks)

(c) $y=\dfrac{\ln\left(x^2+3x\right)}{x}$.

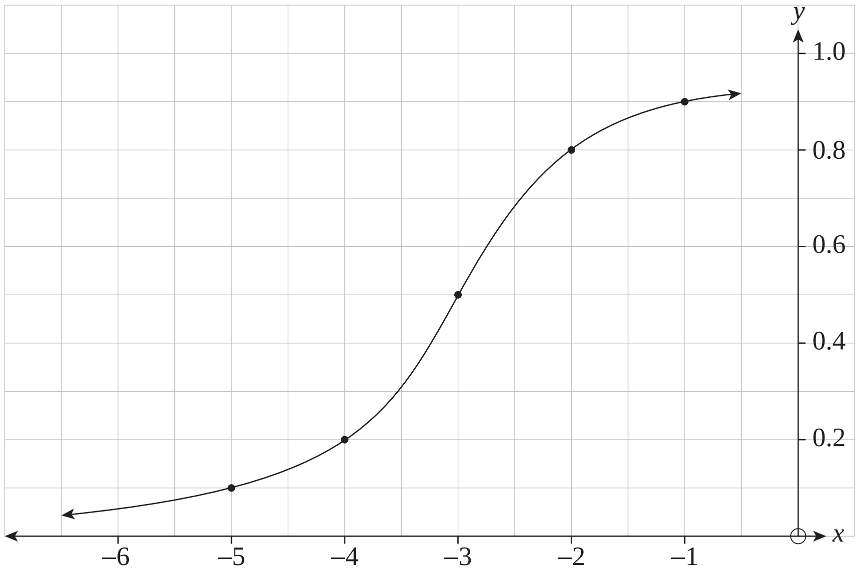
(3 marks)

Question 2 (5 marks)

Consider the table of values below.

x	–5	–4	–3	–2	–1
$f(x)$	0.1	0.2	0.5	0.8	0.9

The function $f(x)$ is continuous and increasing for $-5 \le x \le -1$. The graph of $y = f(x)$ is shown below.

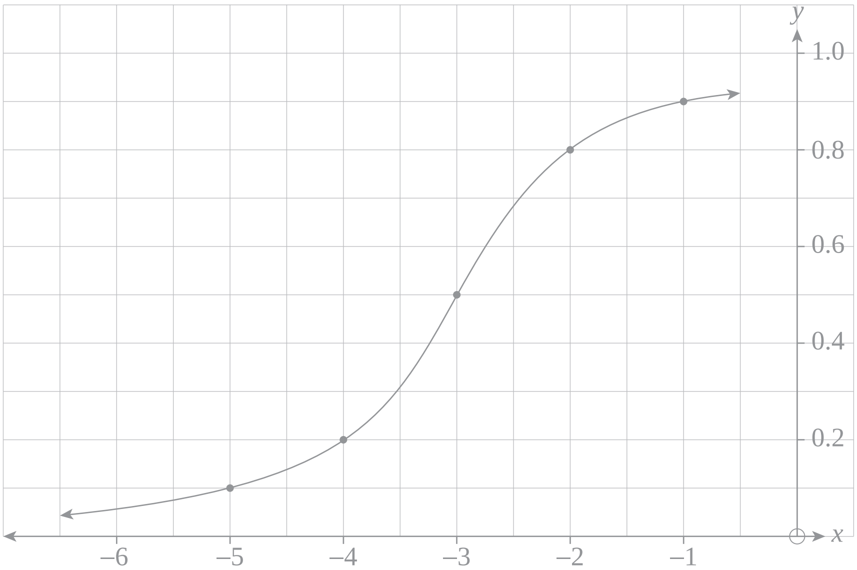


An *over*estimate for the definite integral $\int_{-5}^{-1} f(x)\,dx$ was calculated, using two rectangles. The result of the overestimate is

$$2(0.5)+2(0.9)=2.8.$$

(a) On the graph above, draw the rectangles used to calculate this overestimate. (1 mark)

You may use the spare graph provided below when answering parts (b) and (c); however, you will not earn any marks by doing so.



(b) Calculate an *under*estimate for the definite integral $\int_{-5}^{-1} f(x)\,\mathrm{d}x$, using two rectangles of equal width.

(2 marks)

(c) Calculate an improved underestimate for the definite integral $\int_{-5}^{-1} f(x)\,\mathrm{d}x$, using four rectangles of equal width.

(2 marks)

Question 3 (12 marks)

(a) Consider the discrete random variable X with possible values $\{x_1, x_2, x_3, x_4\}$ and corresponding probabilities $\{p_1, p_2, p_3, p_4\}$ such that $\Pr(X = x_i) = p_i$.

(i) What condition must *each* of p_1, p_2, p_3, and p_4 satisfy?

(1 mark)

(ii) What condition must p_1, p_2, p_3, and p_4 *collectively* satisfy?

(1 mark)

(b) Consider the following probability distribution of the discrete random variable X:

x	1	2	3	4
$\Pr(X = x)$	$\frac{k}{4}$	k^2	k	$\frac{k}{4}$

(i) Show that $2k^2 + 3k - 2 = 0$.

(2 marks)

(ii) Find the value of k for this distribution.

(2 marks)

(iii) Complete the table below, using the value of k that you found in part (b)(ii).

x	1	2	3	4
$\Pr(X = x)$				

(1 mark)

(iv) Find the *exact* value of μ_X.

(2 marks)

(v) Find the *exact* value of σ_X.

(3 marks)

Question 4 (7 marks)

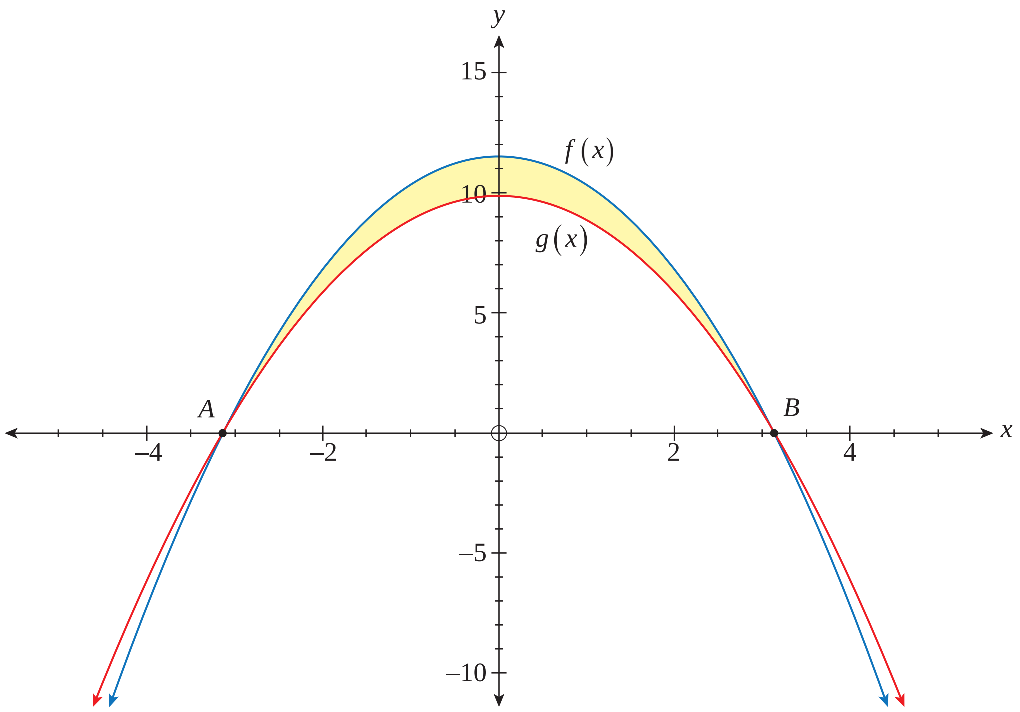
Consider the two functions $f(x)$ and $g(x)$ where

$$f(x) = \sqrt{3}\cos\frac{x}{2} - x^2 + \pi^2$$

and

$$g(x) = -x^2 + \pi^2.$$

These two functions form the boundaries of a shape, as shown shaded on the graph below.



(a) Two intersection points of $f(x)$ and $g(x)$, marked A and B, lie on the x-axis.

Find the *exact* coordinates of points A and B.

(3 marks)

(b) Hence or otherwise, write an integral expression for the area of the shaded shape.

(1 mark)

(c) Hence find the *exact* area of the shaded shape.

(3 marks)

Question 5 (12 marks)

Consider the function $f(x) = \ln(3x+8)$.

(a) For what values of x is $f(x)$ undefined?

(1 mark)

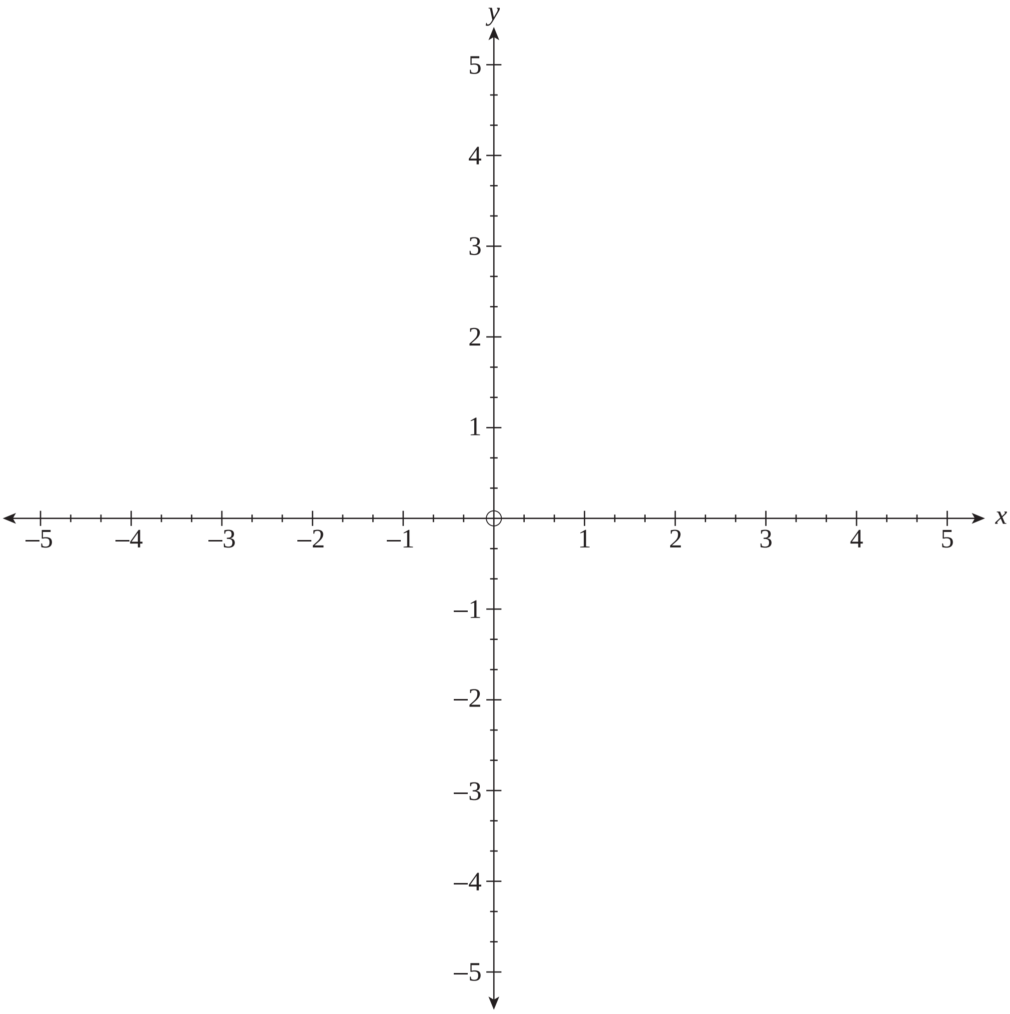
(b) Using algebra, solve the equation $f(x) = 0$.

(2 marks)

(c) Evaluate $f(0)$.

(1 mark)

(d) On the axes below, sketch the graph of $y = f(x)$, clearly showing and labelling the information found in parts (a), (b), and (c).



(3 marks)

Now consider the function $g(x)=\ln(bx+c)$, where $b>0$ and $c>0$.

(e) Find the equation of the asymptote.

(2 marks)

(f) Find the coordinates of the x-intercept.

(1 mark)

(g) Find the slope of the tangent to $g(x)$ at $x=0$.

(2 marks)

Question 6 (9 marks)

Let X be a continuous random variable with probability density function $f(x)$.

(a) (i) Which *one* of the following statements is true? Tick the appropriate box.

$f(x) \geq 0$ ☐

$0 \leq f(x) \leq 1$ ☐

$f(x)$ can take any real value ☐

(1 mark)

(ii) Which *one* of the following statements is true? Tick the appropriate box.

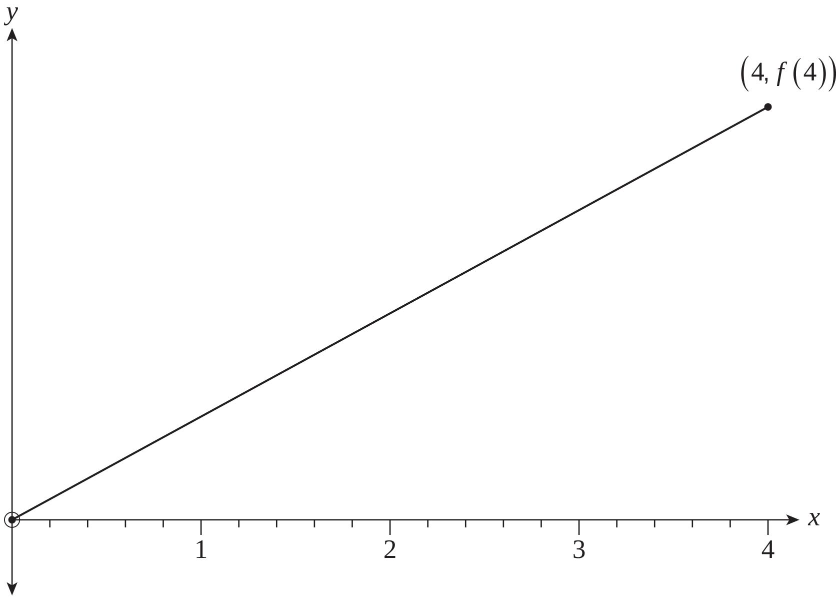
$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0$ ☐

$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ ☐

$\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ could be any real value ☐

(1 mark)

(b) The graph below shows a straight line that represents the probability density function $f(x)$ defined for $0 \le x \le 4$.



(i) Show that $f(x) = \frac{1}{8}x$.

(3 marks)

(ii) Write an integral expression for the mean (μ_X) of the continuous random variable X.

(1 mark)

(iii) Evaluate your expression to determine μ_X.

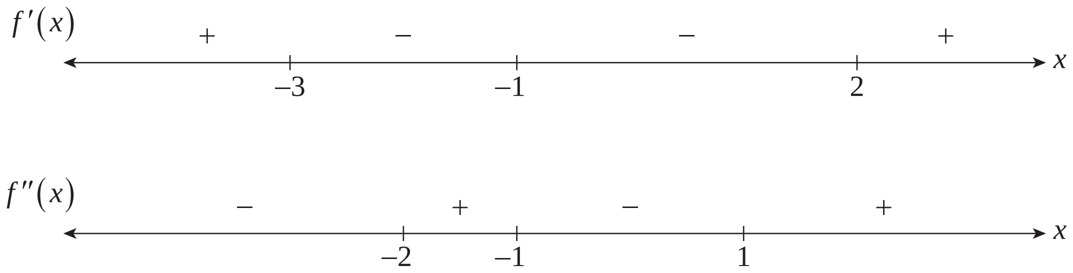
(1 mark)

(iv) Find $\Pr\left(X \geq \mu_X\right)$.

(2 marks)

Question 7 (9 marks)

The sign diagrams below relate to the first derivative, $f'(x)$, and the second derivative, $f''(x)$, of the function $f(x)$.



(a) For what values of x is $f(x)$ decreasing?

(1 mark)

(b) For what values of x is the graph of $y = f(x)$ concave?

(2 marks)

(c) The function $f(x)$ has three stationary points: A, B, and C.

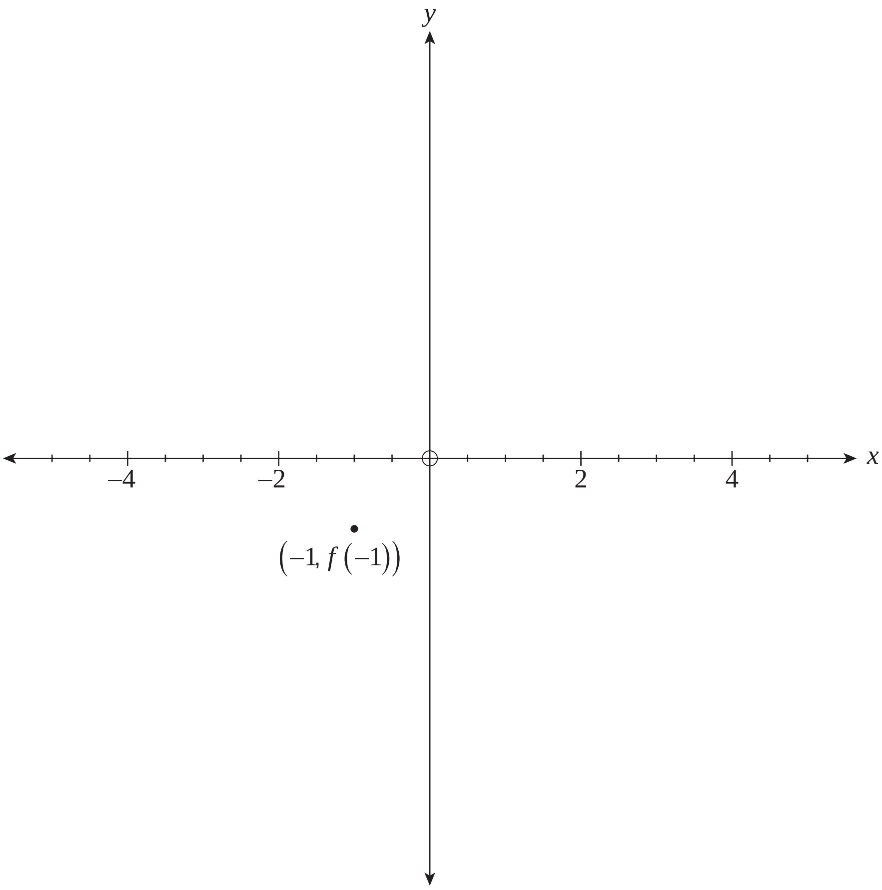
Determine the x-coordinates of points A, B, and C, and classify each point as a local maximum, a local minimum, or a stationary inflection point. Write your answers in the table below.

Stationary point	x-coordinate	Classification
A		
B		
C		

(3 marks)

The function $f(x)$ passes through the point $(-1,\ f(-1))$.

(d) On the axes below, sketch a graph of $y = f(x)$, labelling the stationary points A, B, and C as per the table in part (c).



(3 marks)

Question 8 (7 marks)

(a) Calculate $\int_{-1}^{1} x^3 - 3x + 2 \, dx$. Write your answer in the appropriate space in the table in part (c).

(1 mark)

(b) Calculate $\int_{-2}^{2} x^3 - 3x + 2 \, dx$. Write your answer in the appropriate space in the table in part (c).

(1 mark)

(c) Calculate $\int_{-5}^{5} x^3 - 3x + 2 \, dx$. Write your answer in the appropriate space in the table below.

k	1	2	3	4	5
$\int_{-k}^{k} x^3 - 3x + 2 \, dx$			12	16	

(1 mark)

(d) Make a conjecture for the value of $\int_{-k}^{k} x^3 - 3x + 2 \, dx$ for any value of $k > 0$.

(1 mark)

(e) Prove your conjecture.

(3 marks)

Question 9 (10 marks)

Tri is a student at a school that has a total of 1000 students. Tri has some news that no other student knows. One day, Tri enters the school at 8 am and begins telling some students this news, which then spreads among the students in the school.

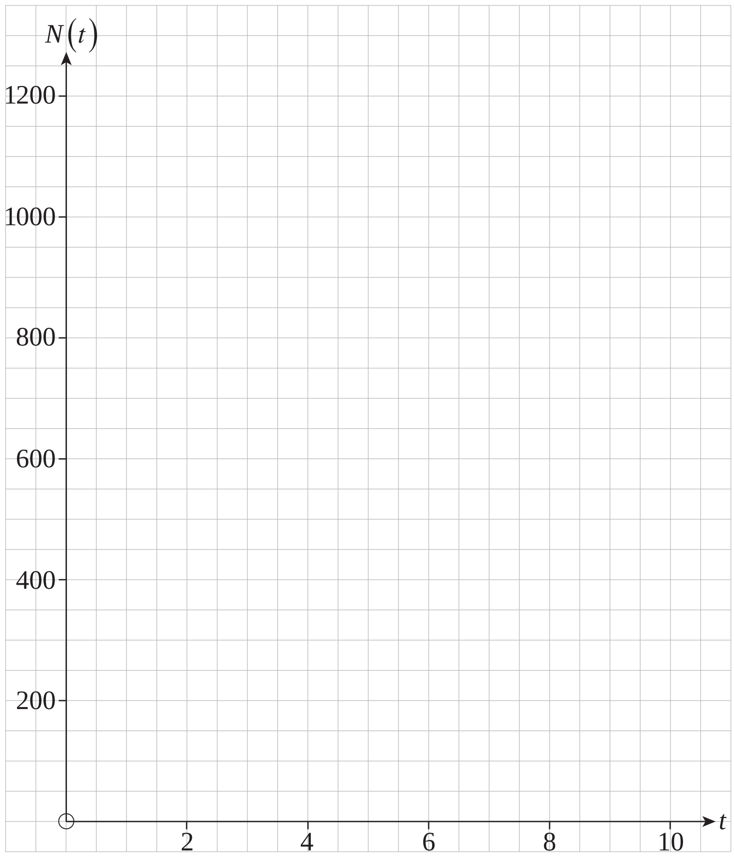
The number, N, of students in this school who have heard Tri's news at time t hours after 8 am can be modelled by the function

$$N(t) = \frac{1000}{1+999e^{-1.386t}}.$$

(a) According to the model, after how many hours will half of the students in this school have heard Tri's news?

(1 mark)

(b) On the axes below, sketch the graph of $N(t)$ against t for $0 \le t \le 10$, clearly showing any intercepts and asymptotes.



(3 marks)

(c) (i) Find an expression for the rate at which Tri's news is spreading among the students in this school, for any value of t.

(3 marks)

(ii) Hence show that $N'(t) \geq 0$ for $t \geq 0$.

(2 marks)

(iii) Comment on the reasonableness of using this model in this context.

(1 mark)

Question 10 (9 marks)

In one particular city, there have been many thefts of student possessions from schools. In an attempt to reduce these thefts, the government has provided new, lockable cupboards to all schools in this city. Each student at each school has been assigned a separate cupboard, and has been instructed to store their possessions in their cupboard and always keep it locked.

The government conducts a survey of school students, to determine how many students always keep their cupboard locked.

From a sample of 320 junior secondary students, the following 95% confidence interval for the proportion of junior secondary students who always keep their cupboard locked is calculated:

$$0.797 \le p \le 0.878.$$

(a) Explain the meaning of this confidence interval.

(2 marks)

(b) (i) What proportion of the students in this sample always keep their cupboard locked?

(1 mark)

(ii) Hence how many of the 320 students in this sample always keep their cupboard locked?

(1 mark)

(c) It is suggested that senior secondary students are less likely than junior secondary students to always keep their cupboard locked. Therefore, the government surveys a sample of 120 senior secondary students, and finds that 74 of these students always keep their cupboard locked.

(i) Calculate a 95% confidence interval for the proportion of senior secondary students who always keep their cupboard locked.

(2 marks)

(ii) Can the government conclude that senior secondary students are less likely than junior secondary students to always keep their cupboard locked? Explain your answer.

(3 marks)

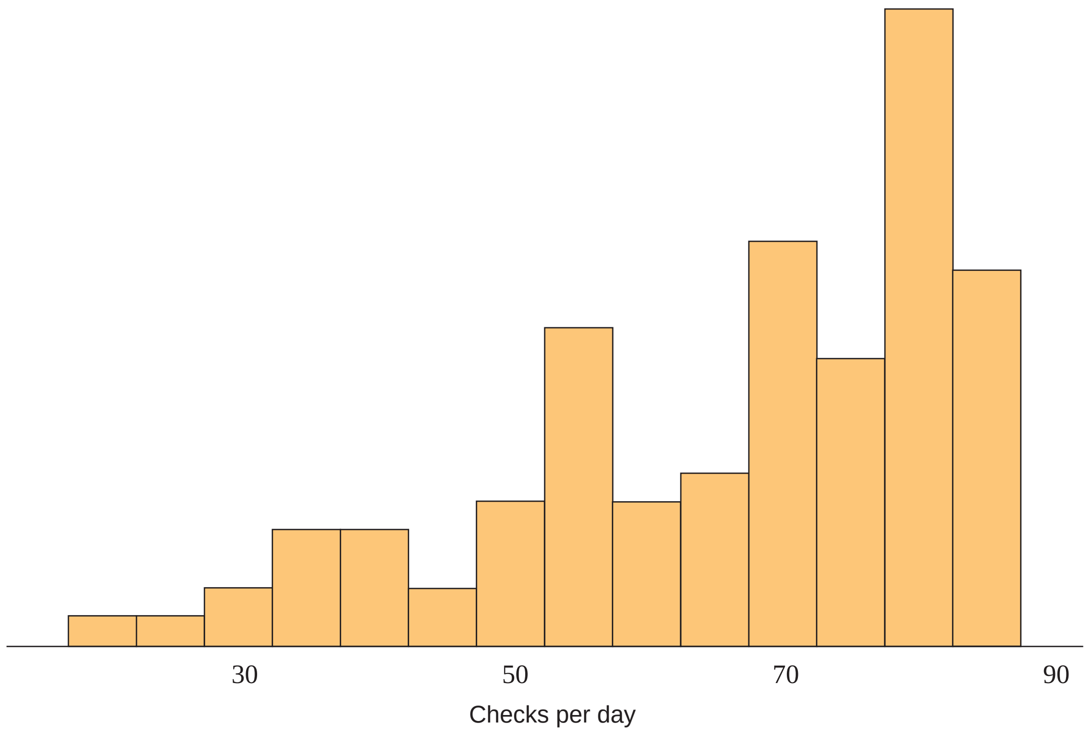
Question 11 (8 marks)

Concerns have been raised over the number of times per day that teenagers check their mobile device (such as a smartphone or a mobile / cell phone).

A study investigated the number of times per day that teenagers check their mobile device. A random sample of 100 teenagers was taken and the number of checks per day (X) was recorded.

The sample mean was found to be $\overline{x} = 62.33$. The sample standard deviation was found to be $s = 19.31$, which is assumed to represent a good estimate for the population standard deviation (σ_X).

The histogram below shows the number of checks per day for this sample of 100 teenagers.

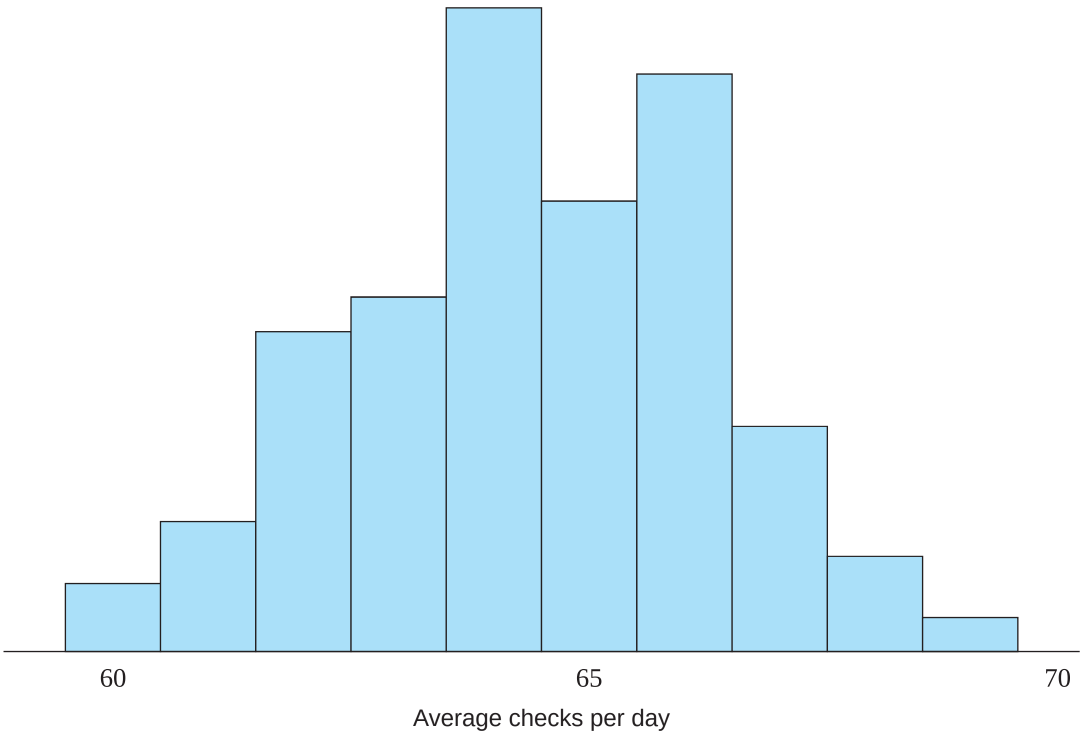


(a) With reference to the histogram above, explain why it may ***not*** be appropriate to use this sample to calculate a confidence interval for the population mean (μ_X).

(1 mark)

Suppose that many random samples of 100 teenagers were taken as part of the study, and the sample mean ($\bar{X}$) for each of these samples was recorded.

The histogram below shows the sample means.



(b) The central limit theorem applies when the sample size is 'sufficiently large'.

With reference to the histogram above, explain why, in this case, the sample size of 100 is sufficiently large.

(1 mark)

(c) Using the sample of 100 teenagers considered in part (a):

(i) calculate a 90% confidence interval for the population mean (μ_X).

(2 marks)

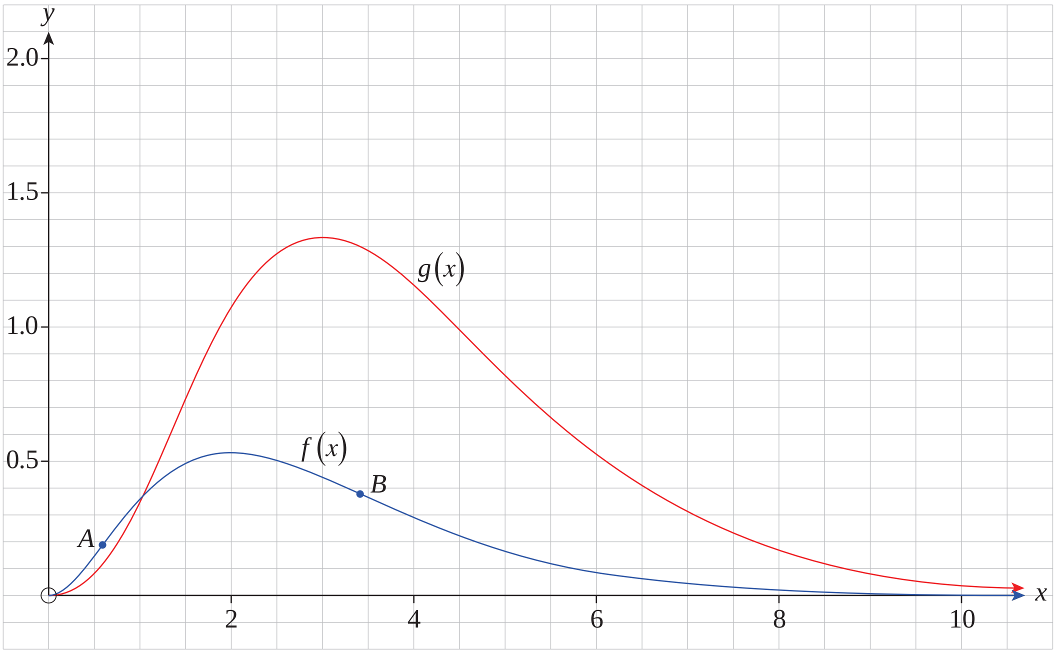
(ii) find the minimum sample size required in order to calculate a 90% confidence interval that has a width of no more than 5 checks per day.

(4 marks)

Question 12 (10 marks)

(a) Consider the functions $f(x) = e^{-x}x^2$ and $g(x) = e^{-x}x^3$.

The graphs of $y = f(x)$ and $y = g(x)$ for $x > 0$ are shown below.



Each graph has two non-stationary points of inflection for $x > 0$.

(i) On the graph of $y = g(x)$, clearly mark the location of the two non-stationary points of inflection. (2 marks)

(ii) Points A and B are the two non-stationary points of inflection on the graph of $y = f(x)$.

Find the coordinates of point A.

(2 marks)

(b) Now consider the general function $h(x)=e^{-x}x^{n}$, for $x>0$ and $n\geq 2$, where n is a real number.

(i) Show that $h''(x)=e^{-x}x^{n-2}\left(x^{2}-2nx+n(n-1)\right)$.

(3 marks)

(ii) Hence show that the graph of $y=h(x)$ for $x>0$ always has two distinct points of inflection.

(3 marks)

Question 13 (9 marks)

In Australia, mobile (cell) phone numbers contain 10 digits, beginning with the digits 04. An example of a mobile phone number is 0402 199 999.

Mobile phone numbers are allocated to telecommunications service providers, for distribution to their customers. This allocation is made according to the table below.

Telecommunications service provider	*Allocated mobile phone numbers*
Optus ('Optus Mobile Pty Limited')	0401 xxx xxx 0402 xxx xxx 0403 xxx xxx
Vodafone ('Vodafone Australia Pty Limited')	0404 xxx xxx 0405 xxx xxx 0406 xxx xxx
Telstra ('Telstra Corporation Limited')	0407 xxx xxx 0408 xxx xxx 0409 xxx xxx 0400 xxx xxx

Source: Data from Australian Communications and Media Authority, *The numbering system*, viewed 18 January 2018, https://www.thenumberingsystem.com.au

A survey company has developed a machine that calls, at random, mobile phone numbers that begin with '040'.

Assume that all mobile phone numbers that begin with 040 are valid, and that customers have not reallocated their mobile phone numbers to another telecommunications service provider.

(a) When the machine calls one mobile phone number:

(i) what is the probability that this mobile phone number is allocated to Telstra?

(1 mark)

(ii) what is the probability that this mobile phone number is ***not*** allocated to Telstra?

(1 mark)

(b) The machine calls batches of five mobile phone numbers.

Let X be the number of these calls that are made to mobile phone numbers that are allocated to Telstra.

(i) State the distribution of X.

(1 mark)

(ii) Within one batch of five mobile phone numbers:

(1) what is the probability that all five calls will be made to mobile phone numbers that are allocated to Telstra?

(1 mark)

(2) what is the probability that at least one call will be made to a mobile phone number that is allocated to Telstra?

(2 marks)

(c) How many mobile phone numbers would the machine need to call in order to have a minimum 99.5% chance of calling at least one mobile phone number that is allocated to Telstra?

(3 marks)

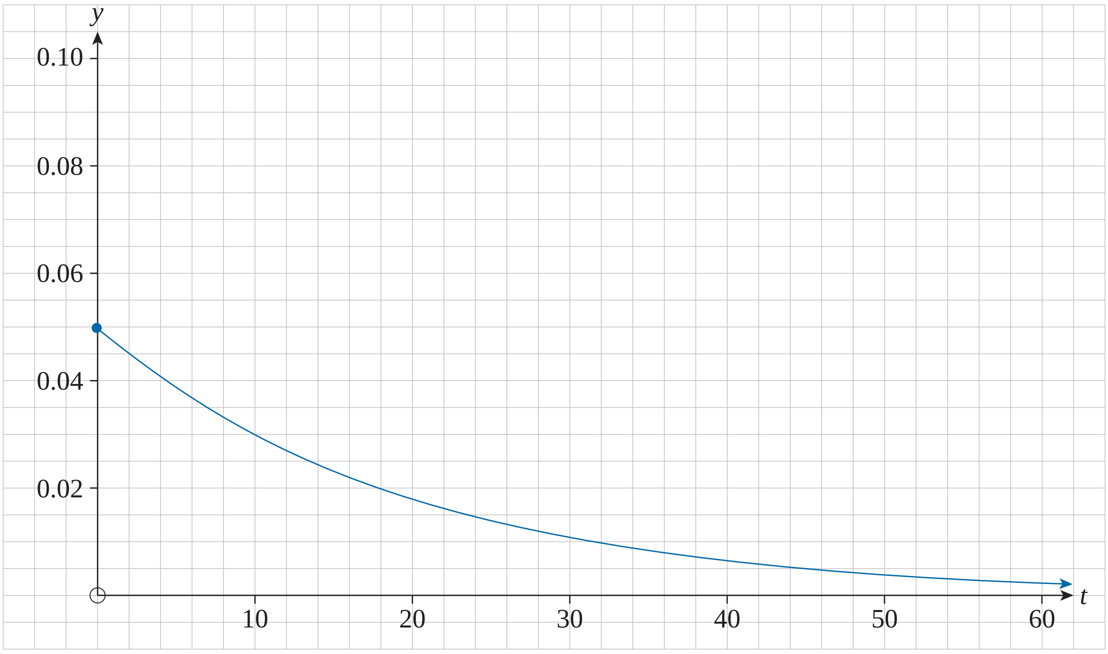
Question 14 (15 marks)

At a sporting event attended by thousands of people, individuals spend time in a queue in order to enter the stadium. The probability that a randomly chosen individual spends time, t, in the entry queue can be modelled by the probability density function

$$f(t) = 0.05e^{-0.05t},$$

where t is measured in minutes and $t \geq 0$.

The graph of $y = f(t)$ is shown below.



(a) (i) Calculate the probability that a randomly chosen individual spends between 0 and 10 minutes in the entry queue.

(2 marks)

(ii) On the graph above, draw a representation of your answer to part (a)(i). (1 mark)

(b) (i) Calculate the probability that a randomly chosen individual spends *less than* 60 minutes in the entry queue.

(1 mark)

(ii) Of 200 individuals who entered the stadium, how many does the model predict spent *more than* 60 minutes in the entry queue?

(2 marks)

(c) Describe *one* limitation of using a function of the form $f(t) = a \times e^{-at}$ to model the time that an individual spends in the entry queue.

(1 mark)

It is a management policy that once inside the stadium, all individuals who spend time in a queue to buy food must be served within 20 minutes. The probability that an individual spends time t in the food queue can be modelled by the probability density function

$$g(t) = B \times e^{-0.05t},$$

where B is a positive value, t is measured in minutes, and $0 \le t \le 20$.

(d) Find the *exact* value of B.

(4 marks)

(e) On the axes on page 13, sketch the graph of $y = g(t)$. (2 marks)

(f) Is the probability of an individual spending between 0 and 10 minutes in the food queue greater than or less than the probability of an individual spending between 0 and 10 minutes in the entry queue?

Give a reason for your answer, but *do not* calculate the probability of spending between 0 and 10 minutes in the food queue.

(2 marks)

Question 15 (16 marks)

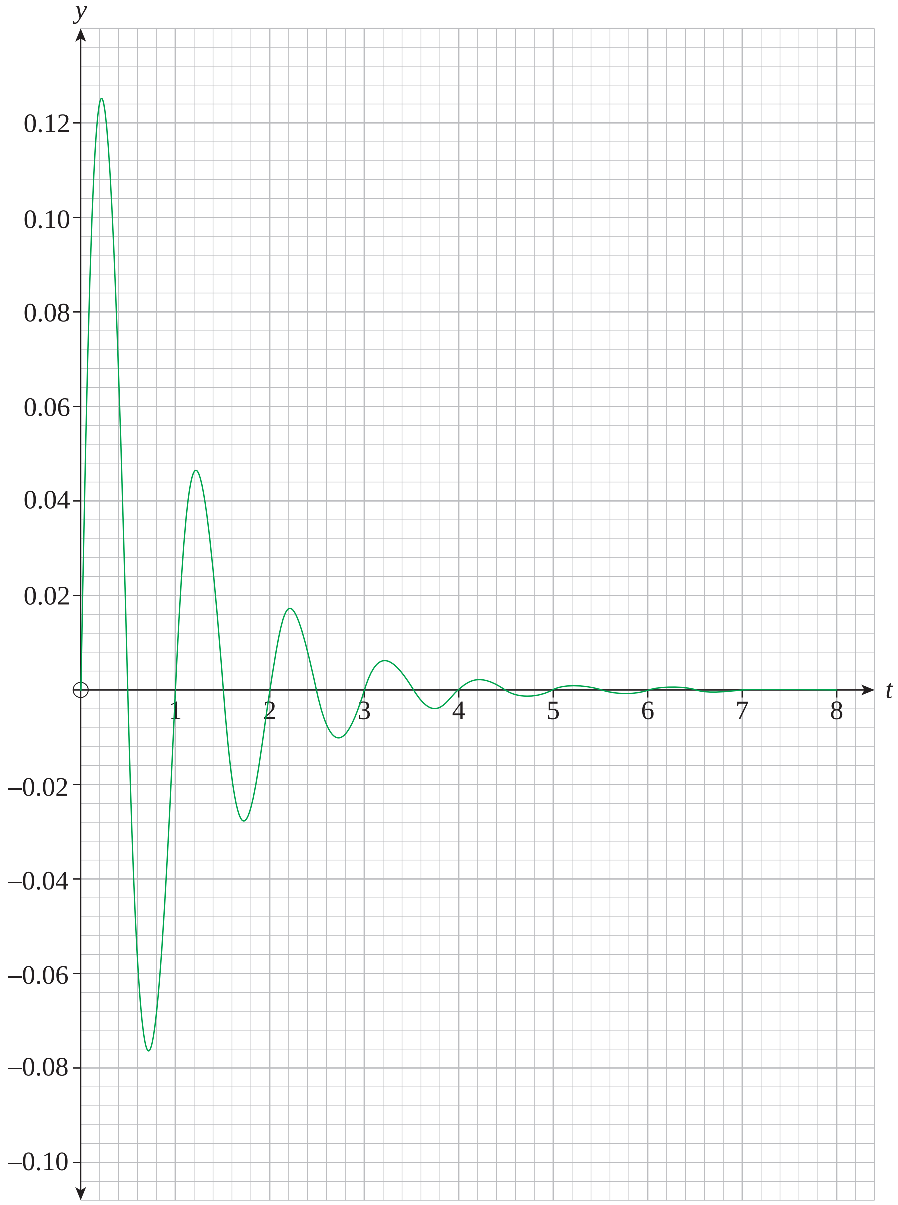
During ‘under-damped oscillation’ an object moves back and forth about a central point, gradually slowing down.

Suppose that an object moves according to the function

$$s(t) = \frac{1}{2\pi} e^{-t} \sin(2\pi t),$$

where positive values of $s(t)$ represent distance in metres to the right of the central point (defined as the origin) and negative values of $s(t)$ represent distance in metres to the left of the origin, at time t seconds, where $0 \le t \le 8$ seconds.

A graph of $y = s(t)$ is shown below.



(a) At $t = 0$, the object is located at the origin.

Show that the object is next located at the origin when $t = 0.5$ seconds.

(2 marks)

(b) (i) Show, by differentiating $s(t)$, that the velocity of the object is given by the function

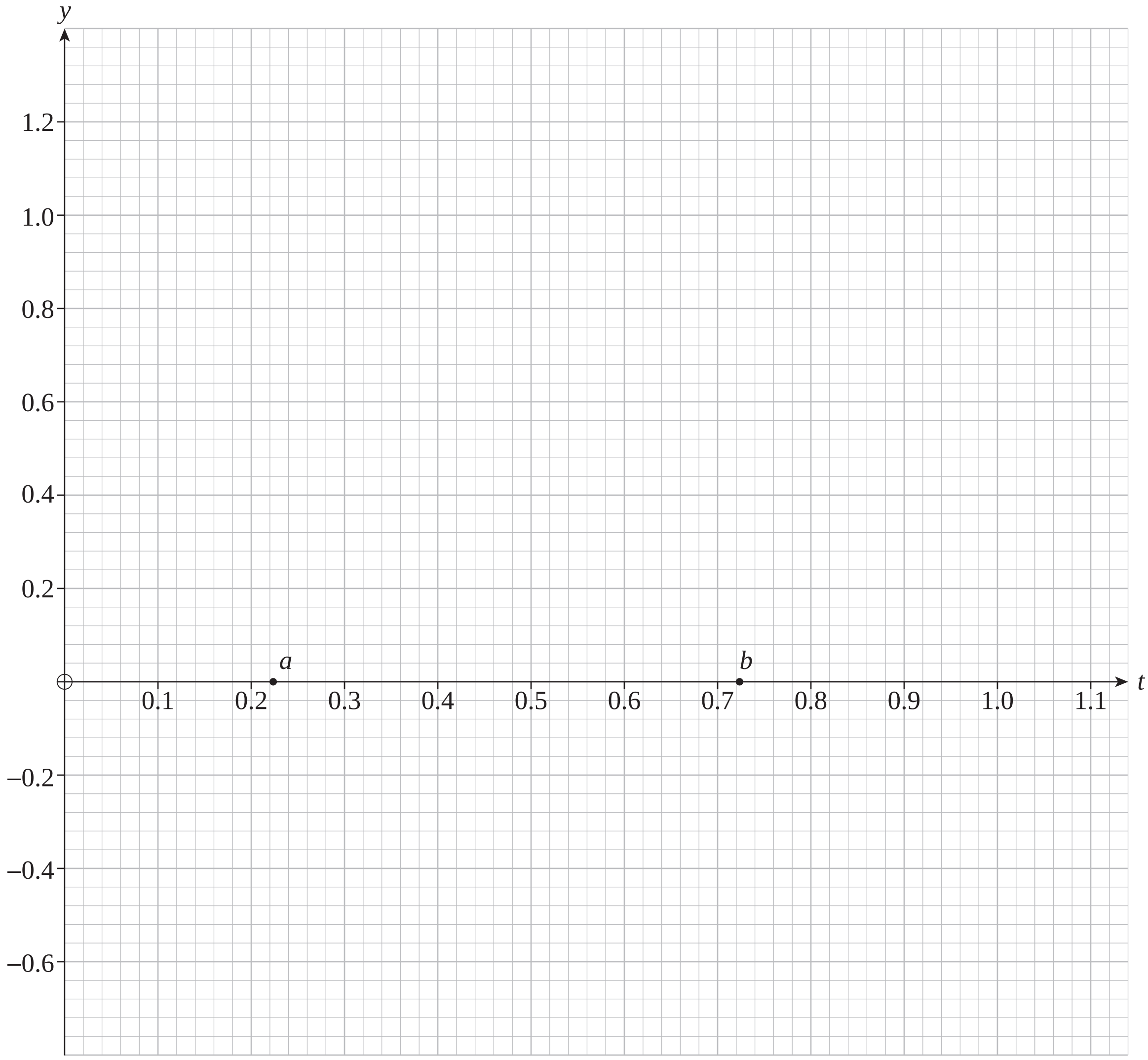
$$v(t) = e^{-t}\left(\cos(2\pi t) - \frac{1}{2\pi}\sin(2\pi t)\right).$$

(2 marks)

(ii) Hence find the initial velocity of the object.

(2 marks)

(c) (i) The intercepts of the graph $y = v(t)$ with the horizontal axis are shown below as points a and b. Sketch the graph of $y = v(t)$.



(2 marks)

(ii) Hence interpret the integral $\int_a^b v(t)\,\mathrm{d}t$, in terms of the movement of the object.

(2 marks)

(iii) Interpret the following expression in terms of the movement of the object:

$$\int_0^a v(t)\,\mathrm{d}t + \int_b^1 v(t)\,\mathrm{d}t.$$

(2 marks)

(iv) Explain why $\int_0^a v(t)\,\mathrm{d}t + \int_b^1 v(t)\,\mathrm{d}t = -\int_a^b v(t)\,\mathrm{d}t$.

(2 marks)

(v) Given $\int_a^b v(t)\,\mathrm{d}t = -0.2017$, find the total distance travelled by the object during the interval $0 \le t \le 1$.

(2 marks)